Non-local 2D Generalized Yang-Mills theories on arbitrary surfaces with boundary

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Abstract

The non-local generalized two dimensional Yang Mills theories on an arbitrary orientable and non-orientable surfaces with boundaries is studied. We obtain the effective action of these theories for the case which the gauge group is near the identity, $U \simeq I$. Furthermore, by obtaining the effective action at the large-N limit, it is shown that the phase structure of these theories is the same as that obtain for these theories on orientable and non-orientable surface without boundaries. It is seen that the ϕ^2 model of these theories on an arbitrary orientable and non-orientable surfaces with boundaries have third order phase transition only on $g = 0$ and $r = 1$ surfaces, with modified area $\tilde{A} + \mathcal{A}/2$ for orientable and $\bar{A} + \mathcal{A}$ for non-orientable surfaces respectivly.

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1 Introduction

The **BF** theory is defined on a Riemann surface by the Lagrangian $i\mathrm{tr}(BF) + \mathrm{tr}(B^2)$, which is equivalent to the Yang- Mills theory. This theory has certain properties such as invariance under area preserving diffeomorphisms and lack of any propagating degrees of freedom [4]. These properties are also shared by a large class of theories, called the generalized two dimensional Yang-Mills (gYM_2's) theories. These theories, however, are defined by replacing an arbitrary class function of B instead of $\mathrm{tr}(B^2)$ [10]. Several aspect of this theories such as, partition function, generating functional and large -N limit on an arbitrary two dimensional orientable and non-orientable surfaces has been discussed in [11-17]. There is another way to generalize YM_2 and gYM_2 and that is to use a non-local action for the auxiliary field, leading to the so-called non-local YM_2 (nlYM_2) and non-local gYM_2(nlgYM_2) theories, respectively [18]. Several aspects of nlYM_2 and nlgYM_2, such as, classical behavior, wave function, partition function, generating functional, and also large-N limit of it, have been studied on orientable and non-orientable surfaces in [18-21]. In all of these theories, the solution appear as some infinite summations over the irreducible representations of the gauge group. In the large - N limit, however, these summations are replaced by suitable path integrals over continuous parameters characterizing the Young tableaux, and saddle-point analysis shows that the only significant representation is the classical one, which minimizes some effective action. This continuous parameters characterizing the representation is a constrained, as the length of the rows of the Young tableau is non-increasing. So for small values of the surface area, the classical solution satisfies the constraint; for large values of the surface area, it dose not. Therefore the dominating representation is not the one, which minimizes the effective action. This introduces a phase transition between these two regime. It can be seen that the $\phi^2 + \frac{2\alpha}{3}\phi^3$ and all order of ϕ^{2k} models of nlgYM_2 for orientable surfaces with no boundaries have a third order phase transition on orientable surfaces with $g \neq 1$ and $g = 0$ respectively [20]. Also, for non-

orientable surfaces without boundary the all order of ϕ^{2k} models of nlgYM_2 have a third order phase transition only on projective plan(RP^2), and the $\phi^2 + \frac{2\alpha}{3}\phi^{2k+1}$ $(k = 0, 1, 2, \ldots)$ model of nlgYM_2 on non-orientable surfaces without boundary has the same phase structure of non-local two dimensional Yang-Mills theory [21]. Furthermore, some aspects of Yang-Mills theory(local and non-local) have been studied for surfaces with boundary in [22, 23 24]. The large-N limit of YM_2 on cylinder and on a sphere with three holes have been studied in [22]. The large gauge group of the non-local YM_2 and generalized YM_2 on a cylinder have been studied in [23], and the authors of [24] have studied the large-N limit of the two dimensional $U(N)$ Yang-Mills theory on an arbitrary orientable compact surface with boundaries. It is seen that, in these cases, for each boundary, the character if the gauge filed corresponding to that boundary appears in the expression of the partition function.

The scheme of this paper is the following. In sect.2, I obtain the partition function of the non-local generalized two dimensional Yang-Mills theory for the case which the gauge group is near the identity. In sect. 3, I study the large-N limit of nlgYM_2 on an orientable and non-orientable surfaces with boundaries. It will be shown that the order of phase transition for ϕ^2 model, is 3 with modified area $\tilde{A} + \mathcal{A}/\in$ and $\bar{A} + \mathcal{A}$ for orientable and non-orientable surfaces respectively.

2 The partition function of nlgYM_2 on surfaces with boundaries

In [18], a non-local generalized two dimensional Yang-Mills (nlgYM_2's) theories was defined as:

$$e^S := \int DB \exp\left\{ \int i\mathrm{tr}(BF)d\mu + \omega\left[\int \Lambda(B)d\mu \right] \right\}, \tag{1}$$

where B is an auxiliary field at the adjoint representation of gauge group, F is the field strength, $d\mu$ is the invariant measure of the surface; $d\mu := \frac{1}{2}\epsilon_{\mu\nu}dx^\mu dx^\nu$ and Λ is a similarity-invariant function. It was further shown that the partition function for this theory for an

arbitrary surface, $\Sigma_{g,n,q}$, with area A and n boundaries is given by the exact formula [18, 19] as:

$$Z(U_1,\ldots,U_n;A(\Sigma_{g,n,q}))=\sum_R h_R^q d^{(2-2g-q)}\prod_{i=1}^n\left(\frac{\chi(U_i)}{d_R}\right)\exp\left\{\omega[-A(\Sigma_{g,n,q})C_\Lambda(R)]\right\}. \quad (2)$$

Here $\Sigma_{g,n,q}$ is a surface containing g handles and q projective planes. R's label the irreducible representation of the gauge group, d_R is the dimension of the representation R, $C_\Lambda(R)$ is a linear function of Casimirs of gauge group in the R representation and $\chi_R(U)$ is the gauge group character. It is clearly seen from (2), corresponding to each boundary a factor $\chi_R(U_i)/d_R$ appears in the expression of the partition function. We want to obtain the character of the gauge group for the case $U\simeq I$. We assume that the gauge group is simple. A group element is characterized by some parameters such as:

$$U=e^{\theta^\alpha J_\alpha}, \quad (3)$$

where J_α's are the generators of the gauge group. For $U\simeq I$ and by expanding the gauge group U up to second order of θ^α's, one can arrive at

$$U=I+\theta^\alpha J_\alpha+\frac{1}{2}\theta^\alpha\theta^\beta J_\alpha J_\beta+\ldots \quad (4)$$

then by using of this fact that the representation of the generators of simple groups are trace less, we obtain

$$\chi(U)=d_R+\frac{1}{2}\theta^\alpha\theta^\beta\chi_R(J_\alpha J_\beta)+\ldots \quad (5)$$

in which for a simple group,

$$\chi_R(J_\alpha J_\beta)=\frac{d_R}{d_U}C_2(R)\mathcal{B}_{\alpha\beta}, \quad (6)$$

where d_U and $\mathcal{B}$ are the dimension and the Killing form of the group, respectively. Therefore for a simple group up to second order of θ^α's, we have

$$\prod_{j=1}^n\frac{\chi_R(U_j)}{d_R}=e^{-\frac{A}{2N}C_2(R)}, \quad (7)$$

where

$$\mathcal{A} = -\frac{N}{d_U}\mathcal{B}_{\alpha\beta}\sum_{j=1}^{n}\theta_j^{\alpha}\theta_j^{\beta}. \tag{8}$$

So by institute (7) in (2), one arrives at

$$Z(U_1,\ldots,U_n;A(\Sigma_{g,n,q})) = \sum_R h_R^q d^{(2-2g-q)} \exp\Big\{-\frac{\mathcal{A}}{2N}C_2(R) + \omega[-A(\Sigma_{g,n,q})C_\Lambda(R)]\Big\}. \tag{9}$$

Note that the exponent in (9) consists of two parts. The first part coming from the characters and the second part depends on the non-locality term of the action of the non-local generalized two dimensional Yang-Mills theory (1). Therefore for orientable surfaces $q = 0$, and the partition function on an orientable surface is

$$Z(U_1,\ldots,U_n;A(\Sigma_{g,n}) = \sum_R d^{(2-2g)} \exp\Big\{-\frac{\mathcal{A}}{2N}C_2(R) + \omega[-A(\Sigma_{g,n})C_\Lambda(R)]\Big\}, \tag{10}$$

but for non-orientable surface $q \neq 0$ and h_R is defined as

$$h_R := \int \chi_R(U^2)dU. \tag{11}$$

Here h_R is zero unless the representation R is self- conjugate. In this case, the representation has an invariant bilinear form. Then, $h_R = +1(-1)$if this form is symmetric(antisymmetric), so the partition function on non-orientable surfaces is

$$Z(U_1,\ldots,U_n;A(\Sigma_{g,n,q}) = \sum_{R=\bar{R}} d^{(2-2g-q)} \exp\Big\{-\frac{\mathcal{A}}{2N}C_2(R) + \omega[-A(\Sigma_{g,n,q})C_\Lambda(R)]\Big\}, \tag{12}$$

where the summation is only over self-conjugate representation of the gauge group.

3 Large-N limit

3.1 Large-N limit on orientable surfaces

In the large-N limit of the $U(N)$ gauge group, we assume that C_Λ is linear function of the rescaled Casimirs of the gauge group, in which the kth rescaled Casimir of the gauge group $U(N)$ is as following

$$\tilde{C}_k(R) =: \frac{1}{N^{k+1}}\sum_{i=1}^{N}(l_i + N - i)^k, \tag{13}$$

where l_i's characterize the representation of the gauge group which satisfy $l_i \geq l_j (i \geq j)$ and it is found that

$$d_R = \prod_{1 \leq i \leq j \leq N} \left(1 + \frac{l_i - l_j}{j - i} \right). \tag{14}$$

One can define a function W as

$$-N^2 W \left[A(\Sigma) \sum_k a_k \tilde{C}_k(R) \right] := w[-AC_\Lambda(R)], \tag{15}$$

so in the large-N limit, (10) is written as

$$Z = \int D\phi(x) \exp \left\{ - N^2 \Big(S_0[\phi] + S_1[\phi] \Big) \right\}. \tag{16}$$

where

$$\phi \;\; := \;\; \frac{i - n_i - N}{N}, \tag{17}$$

$$x \;\; := \;\; \frac{i}{N}, \tag{18}$$

and

$$S_0[\phi] = -\frac{\mathcal{A}}{2} \int_0^1 \phi^2(x) dx, \tag{19}$$

where is coming from the characters and boundaries. Also

$$S_1[\phi] = W \left(A \int_0^1 G[\phi(x)] dx \right) - (1 - g)) \int_0^1 dx \int_0^1 dy \log |\phi(x) - \phi(y)|, \tag{20}$$

$$G[\phi(x)] = \sum_k (-1)^k a_k \phi^2(x). \tag{21}$$

$S_1[\phi(x)]$ is the same action which appear on non-local generalized two dimensional Yang-Mills theory on closed surface. It is obvious that the phase structure of this term is equivalent to the phase structure of the following action [20]

$$S_1'[\phi] = \tilde{A} \int_0^1 G[\phi(x)] dx - (1 - g)) \int_0^1 dx \int_0^1 dy \log |\phi(x) - \phi(y)|, \tag{22}$$

with

$$\tilde{A} = 2AW' \left(A \int_0^1 G[\phi(x)] dx \right). \tag{23}$$

Therefore we can write the effective action which explain the phase structures of this theory as

$$S_{eff}[\phi(x)] = \int_0^1 \mathcal{L}[\phi(x)]dx + (g-1)) \int_0^1 dx \int_0^1 dy \log|\phi(x) - \phi(y)|, \tag{24}$$

where

$$\mathcal{L}[\phi(x)] = \frac{\mathcal{A}}{2}\phi^2 + \tilde{A}G[\phi]. \tag{25}$$

It is remarkable that S_{eff} is nearly the same action for the corresponding *local* generalized Yang-Mills Theory. The differences are an additional term which coming from characters of the gauge group and the existance of $\tilde{A}$ instead of A.

3.2 Large-N limit on non-orientable surfaces

Starting from (12), note that the sum in (12) is only over self conjugate representations. So by applying this additional constraint to the sum and then on continuum variable in the large N limit and repeat the same approach in the previous subsection, we obtain

$$Z = \int D\psi(x) \exp\left\{ - N^2\Big(S_0[\psi] + S_1[\psi]\Big)\right\}, \tag{26}$$

where the function $\psi(x)$ being defined on the interval $[0, 1/2]$, in which $\psi(1/2) = 0$, and

$$S_0[\psi(x)] = \mathcal{A}\int_0^{\frac{1}{2}} \psi^2(x)dx, \tag{27}$$

which coming from characters of the gauge group and

$$S_1[\psi] = W\left(2A\int_0^{1/2} G[\psi(x)]dx\right) - 2(1-(g+q/2))\int_0^{1/2} dx \int_0^{1/2} dy \log|\psi^2(x) - \psi^2(y)|. \tag{28}$$

It is obvious that the phase structure of this term is equivalent to the phase structure of the following action [21],

$$S_0'[\psi] = \bar{A}\int_0^{1/2} G[\psi(x)]dx - 2(1-(g+q/2))\int_0^{1/2} dx \int_0^{1/2} dy \log|\psi^2(x) - \psi^2(y)|, \tag{29}$$

where

$$\bar{A} = 4AW'\left(2A\int_0^{1/2} G[\psi(x)]dx\right).$$

So the effective action which explain the phase structures of this theory(nlgYM$_2$) on a non-orientable surface is

$$S^{no}_{eff}[\psi] = \int_0^{1/2} \mathcal{L}^{no}[\psi(x)]dx - 2(1-(g+q/2)) \int_0^{1/2} dx \int_0^{1/2} dy \log|\psi^2(x) - \psi^2(y)|, \quad (30)$$

where

$$\mathcal{L}^{no}[\psi(x)] = \mathcal{A}\psi^2(x) + \bar{A}G[\psi(x)]. \quad (31)$$

It is seen that this partition function is equal to the partition function on a non-orientable surface with modified area $\bar{A}$, genus g, q copies of projective plane (RP^2), and without boundaries [21]. So that we can obtain the phase structures of this theory with the same procedure in [21].

4 The $G[\phi] = \frac{1}{2}\phi^2$ model

This model defines a non-local Yang-Mills theory. In this case the effective action on an orientable and non-orientable surfaces are

$$S^0_{eff}[\phi(x)] = (\frac{\mathcal{A}}{2} + \tilde{A}) \int_0^1 \phi^2(x)dx + (g-1)) \int_0^1 dx \int_0^1 dy \log|\phi(x) - \phi(y)|, \quad (32)$$

$$S^{no}_{eff}[\psi] = (\mathcal{A} + \bar{A}) \int_0^{1/2} \psi(x)dx - 2(1-(g+q/2)) \int_0^{1/2} dx \int_0^{1/2} dy \log|\psi^2(x) - \psi^2(y)|, \quad (33)$$

respectively. It is shown that this theory has third order phase transition on an orientable surface with boundary , $g = 0$ and modified area $(\frac{\mathcal{A}}{2} + \tilde{A})$. Also this model has third order phase transition on non-orientable surface with boundary, $g = 0, q = 1$ and modified area $(\mathcal{A} + \bar{A})$.

5 conclusion

I study the non-local generalized two dimensional $U(N)$ Yang-Mills (nlgYM$_2$) theories on an arbitrary orientable and non-orientable surface with boundaries. We obtain the action of

this theory for the case which the holonomies of the gauge field on boundaries are near the identity, $U \simeq I$, on arbitrary surface. By obtaining the effective action of these theories at the large-N limit it is shown that, the effective action of it is the same as that obtain on an arbitrary orientable and non-orientable surface without boundaries with the same genus and projective plan but with modified area and an addition term which coming from characters of the gauge group. It is seen that the $\mathcal{A}$ term is a function of holonomies of the gauge field only, in which for orientable and non-orientable surfaces with the same boundaries is equal. Furthermore, for $G[\phi] = \phi^{2k+1}(k \in Z)$ the functional $\mathcal{L}^{no}_{eff} = \mathcal{A}\phi^2$. Therefore the phase structures of $G[\phi] = \phi^{2k+1}(k \in Z)$ models on non-orientable surface is equal to the phase structures of ϕ^2 model of local $2D$ Yang-Mills theory with modified area equal to $\mathcal{A}$.

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